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A method to investigate primality

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A method to investigate primality

The method determines the smallest odd prime factor of a number N by testing the remainders left after division by the successive odd numbers 3, 5, ... $f_m - 2$, f_m; here f_m is the largest odd number not exceeding $N^{\frac{1}{2}}$. If none of these remainders vanishes, N is a prime number.

Let f be one of the odd divisors or odd trial divisors.
Remainder r_0 and quotient q_0 are defined by the relations

$$N = r_0 + f.q_0, \qquad 0 \leq r_0 < f.$$

Now q_0 is divided by f + 2, giving

$$q_0 = r_1 + (f + 2).q_1, \quad 0 \leq r_1 < f + 2 .$$

Then q_1 is divided by (f + 4) etc. and this process is continued till a quotient (q_n say) equal to zero is found; r_n is the last remainder in the sequence unequal to zero. After elimination of the q_i we get the relations

$$N = r_0 + f.r_1 + f.(f+2)r_2 + f.(f+2).(f+4)r_3 + \ldots + f.(f+2) \ldots \ldots (f+2n-2).r_n \qquad (1)$$

and

$$0 \leq r_i < f + 2i \qquad (2)$$

Once the sequence r_i is known for a given value of f, it is easy to compute the corresponding sequence r_i^*, defined by the relations (1) and (2) with respect to $f^* = f + 2$, as they are expressed in terms of the r_i by the recurrence relations

$$b_0 = 0, \; r_i^* = r_i - 2(i+1)r_{i+1} - b_i + (f^*+2i)b_{i+1} (i = 0,1,\dots,n) \tag{3}$$

The relation corresponding to (1) is satisfied for arbitrary values of the numbers b_i with $i \geqslant 1$; they are fixed, however, by the relations corresponding to (2)

$$0 \leqslant r_i^* < f^* + 2i \; . \tag{2^*}$$

On account of the inequalities (2) and (2^*) - and $b_0 = 0$ - the b_i satisfy the inequalities

$$0 \leqslant b_i \leqslant 2i \; . \tag{4}$$

We have chosen $b_0 = 0$. Then the relations (3) and (2^*) with $i = 0$ determine r_0^* and b_1; once b_1 is known, (3) and (2^*) with $i = 1$ determine r_1^* and b_2, etc. The process is easily programmed.
0, and the inequalities (2^*) with $i = n$ are always satisfied with $b_{n+1} = 0$, the process terminates with

$$r_n^* = r_n - b_n$$

As soon as $r_n^* = 0$ is found - in that case it can be proved, that $r_{n-1}^* \neq 0$ - the index n, marking the last $r_i \neq 0$ in the sequence, is lowered by 1.

In order to find the smallest odd prime factor of N, the r_i def by (2) and (3) and $f = 3$ are computed. Here the only divisions in the process are carried out. At the same time the initial value of n is found. If N is large, this value may be considerable: for instance $n = 11$ is found for $N \approx 10^{13}$. The amount of work involved in each step is roughly proportional to n^2. Fortunately large initial values of n decrease very rapidly. As soon as $f.(f + 2).(f + 4) > N$, n takes the value 2. This is its minimum value: when $r_n^* = 0$ with $n = 2$ is found $(f^* + 2)^2 > N$ holds and N is a prime number. (If not, we should have found a $r_0 = 0$ earlier a should have stopped there.)

The process still may be speeded up. Let b_n' be the minimum of b_n for fixed n up till a certain moment: then it can be shown that the next b_n satisfies

$$b_n \leqslant b_n' + 1 \; .$$

Let us apply this to the last stage $n = 2$. According to (4) b_2 satisfies $0 \leq b_2 \leq 4$. According to (5), however, the only possible values for b_2 are 0 and 1 as soon as a value $b_2 = 0$ once has been found. This is bound to happen for f ranging (roughly) from $(4N)^{\frac{1}{3}}$ till $(8N)^{\frac{1}{3}}$. In the case $b_2 = 0$ it is apparently unnecessary to test whether $r_2 = 0$ is reached. (If $N \geq 144$, the case $b_n = 0$ with $n = 2$ occurs, before $r_n^* = 0$ with $n = 2$ is found: prime numbers are then always detected in this last stage.)

The less efficient steps of the process for large n (i.e. small f) could be avoided by carrying out divisions for small values of f. (Cf. the method suggested by G.G. Alway, MTAC v. 6,$_1$p. 59-60, where one seems to be obliged to do this for f up to $(8N)^{\frac{1}{3}}$.) However, we strongly advise not to do this.

If the process described above is started at $f = 3$, the whole computation can be checked at the end by inserting the final values of f and r_1 into (1). As all the intermediate results are used in the computation, this check seems satisfactory.

If a double length number N is to be investigated, another argumen can be added: division of N by small f may give a double length quotient, i.e. two divisions (and two multiplications to check) are needed for each f. In our case even only part of the initial n divisions are double length divisions.

If some consecutive odd numbers are to be investigated, the initial divisions are only necessary for the first number N; it is easy to compute the sequence of initial remainders r_1 "representing $N + d$ at $f = 3$" from the corresponding sequence for N at $f = 3$, if d is small.

The process described above has been programmed for the ARMAC (Automatische Rekenmachine van het MAthematisch Centrum).
The speed of this machine is about 2400 operations per second.
A twelve decimal number was identified as the square of a prime in less then 23 minutes.